Supplemental material for "Alternate-wrapped circular distributions"

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1. Simulation results for AWE distribution

Table 1. *Box Plots of the 10000 simulated values of maximum likelihood estimators and Moment estimators for λ for AWE distribution.*

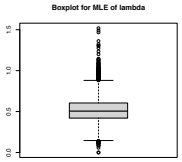
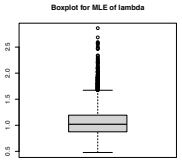
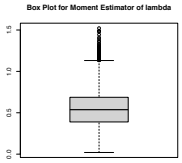
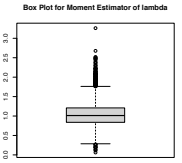
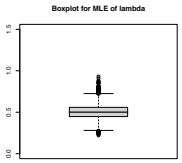
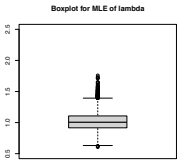
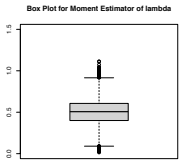
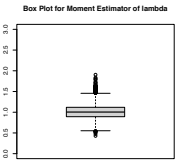
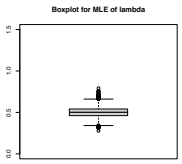
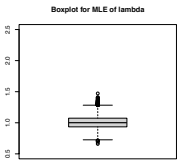
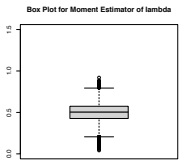
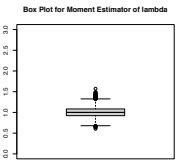
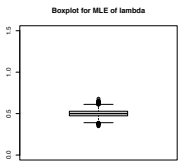
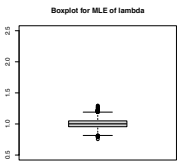
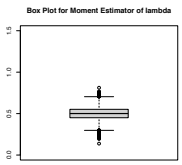
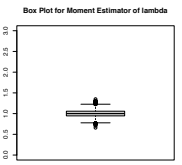
	Box Plots for Maximum Likelihood Estimator ($\hat{\lambda}$)		Box Plots for Moment Estimator ($\tilde{\lambda}$)	
n	0.5	1	0.5	1
20				
50				
100				
200				

Table 2. *Box Plots of the 10000 simulated values of maximum likelihood estimators and Moment estimators for λ for AWE distribution.*

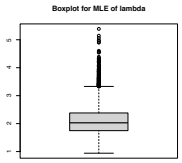
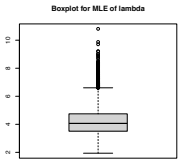
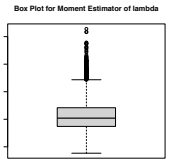
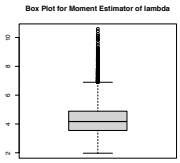
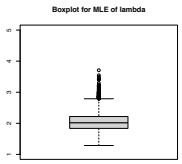
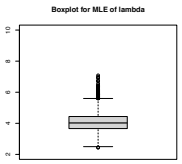
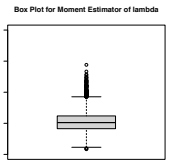
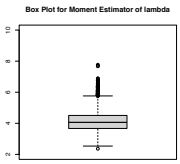
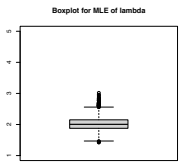
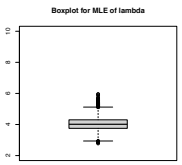
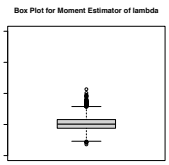
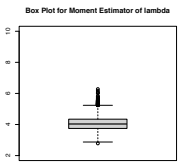
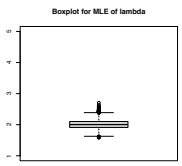
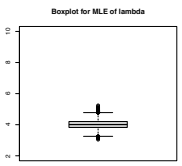
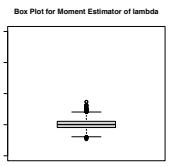
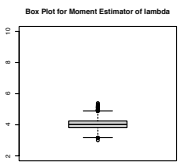
	Box Plots for Maximum Likelihood Estimator ($\hat{\lambda}$)		Box Plots for Moment Estimator ($\tilde{\lambda}$)	
n	2	4	2	4
20				
50				
100				
200				

Table 3. *Fitting of normal distribution to the 10000 simulated values of $\hat{\lambda}$ and $\tilde{\lambda}$ for AWE distribution.*

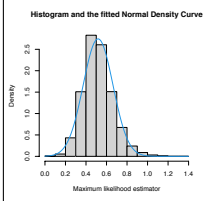
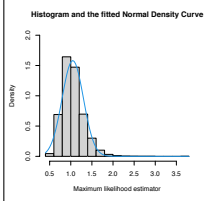
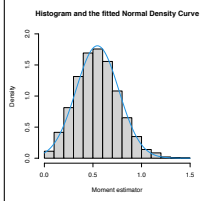
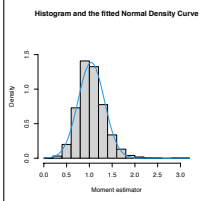
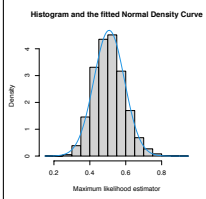
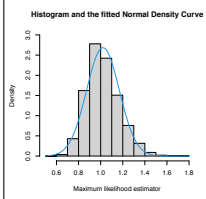
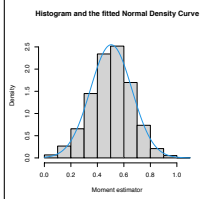
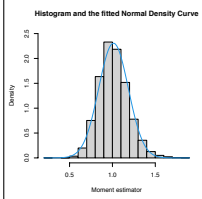
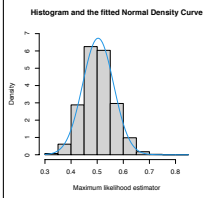
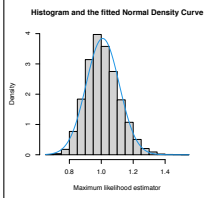
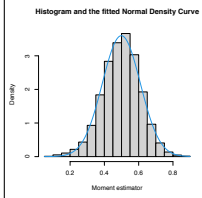
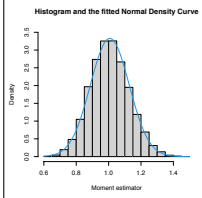
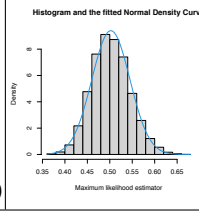
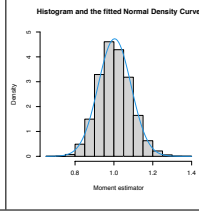
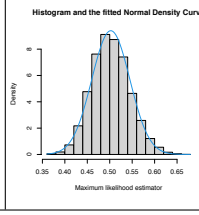
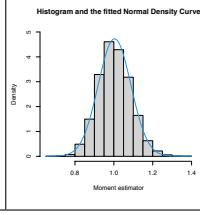
	Fitted normal curve of $\hat{\lambda}$		Fitted normal curve of $\tilde{\lambda}$	
n	0.5	1	0.5	1
20				
50				
100				
200				

Table 4. *Fitting of normal distribution to the 10000 simulated values of $\hat{\lambda}$ and $\tilde{\lambda}$ for AWE distribution.*

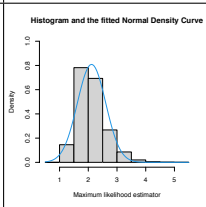
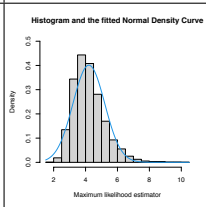
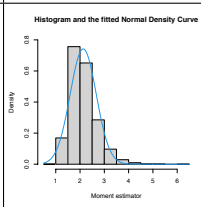
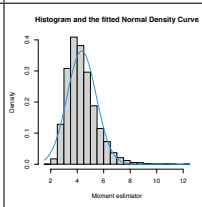
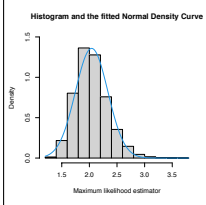
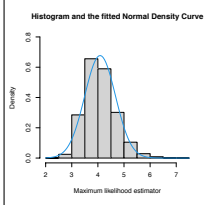
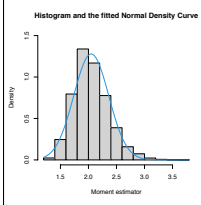
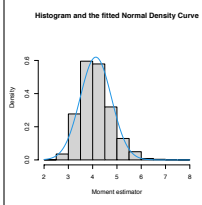
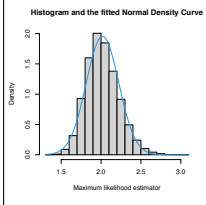
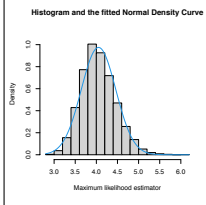
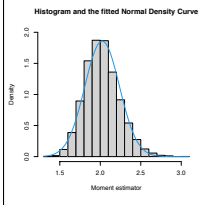
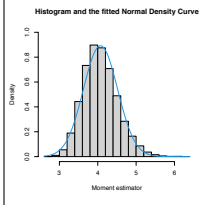
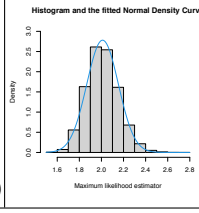
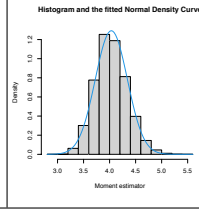
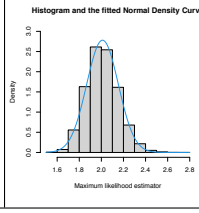
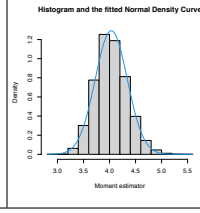
	Fitted normal curve of $\hat{\lambda}$		Fitted normal curve of $\tilde{\lambda}$	
n	2	4	2	4
20				
50				
100				
200				

Table 5. *Chi Square Goodness of fit statistic for MLE and Moment Estimator for AWE distribution.*

	Chi square statistic				Chi square statistic			
n	λ=0.5	λ=1	λ=2	$\lambda = 4$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 2$	$\lambda = 4$
20	6.1938	41.2817	70.3038	143.9720	11.9720	9.4022	90.7871	185.2226
50	0.7590	4.1671	9.8653	17.0929	1.3345	2.5962	10.2162	22.6251
100	0.2685	0.6205	1.8405	5.2971	1.5259	0.2220	2.9434	6.2952
200	0.0867	0.2297	0.4121	1.2120	0.4427	0.1394	0.6079	1.6271

Table 6. *Kernel Density Estimator (KDE) of the density of $\hat{\lambda}$ (MLE) and those of the density of $\tilde{\lambda}$ (moment estimator) based on 10000 simulated values of the estimators of λ for AWE distribution.*

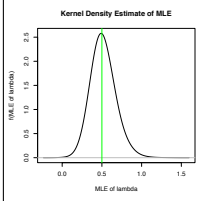
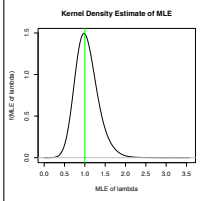
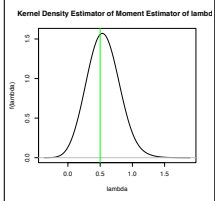
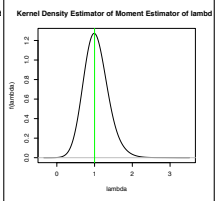
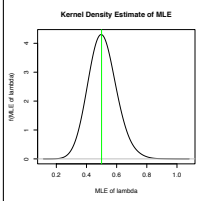
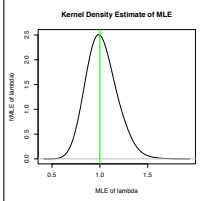
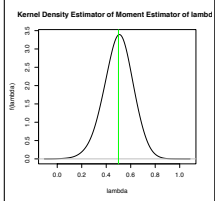
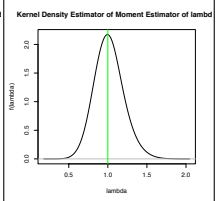
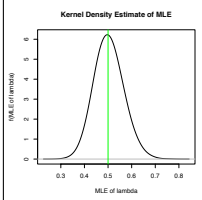
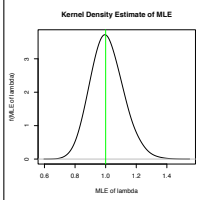
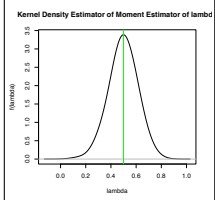
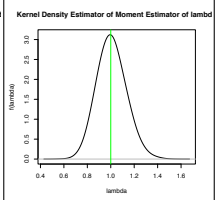
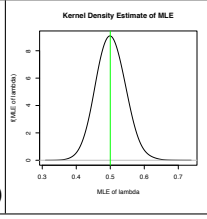
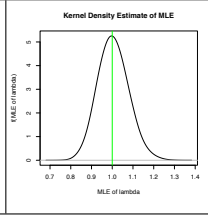
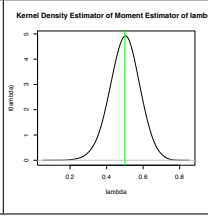
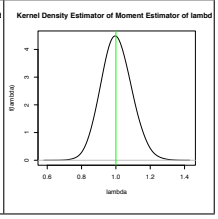
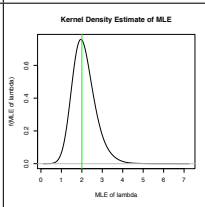
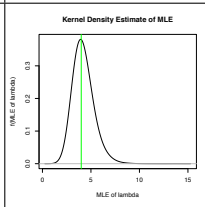
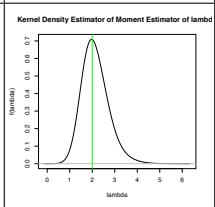
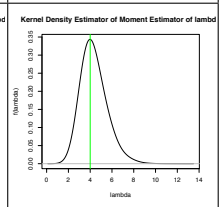
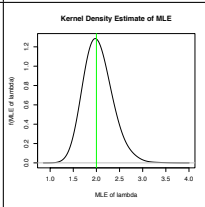
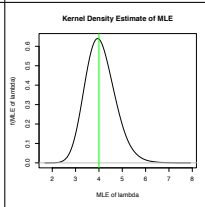
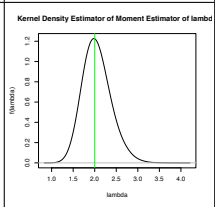
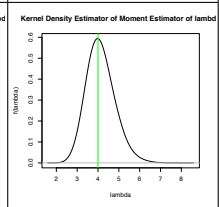
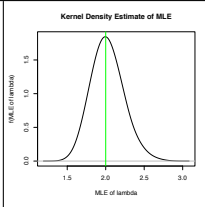
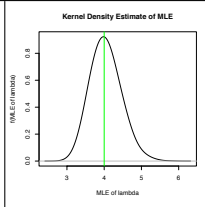
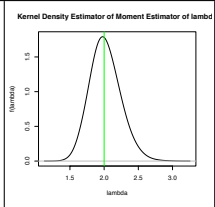
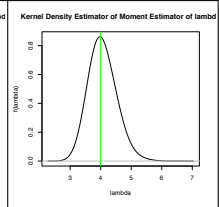
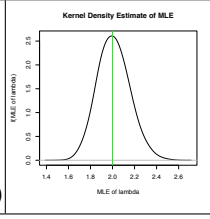
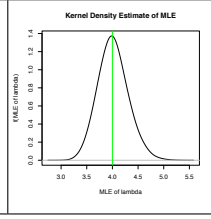
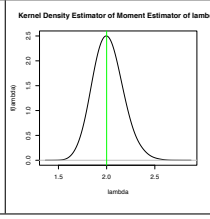
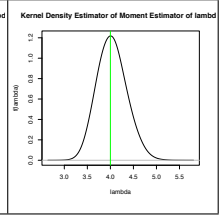
	KDE for the density of $\hat{\lambda}$		KDE for the density of $\tilde{\lambda}$	
n	0.5	1	0.5	1
20				
50				
100				
200				

Table 7. *Kernel Density Estimator (KDE) of the density of $\hat{\lambda}$ (MLE) and those of the density of $\tilde{\lambda}$ (moment estimator) based on 10000 simulated values of the estimators of λ for AWE distribution.*

	KDE for the density of $\hat{\lambda}$		KDE for the density of $\tilde{\lambda}$	
n	2	4	2	4
20				
50				
100				
200				

2. Simulation results for AWN distribution

Table 8. *Average values of bias, MSE and var for $\hat{\mu}$ and $\hat{\sigma}$ based on 10000 samples for AWN distribution.*

	$\sigma = 0.75$						
n	μ	bias($\hat{\mu}$)	MSE($\hat{\mu}$)	var($\hat{\mu}$)	bias($\hat{\sigma}$)	MSE($\hat{\sigma}$)	var($\hat{\sigma}$)
20	$\mu = \pi/2$	0.009590	0.024644	0.024553	-0.044665	0.012347	0.010379
20	$\mu = \pi$	0.001805	0.027219	0.027215	-0.028258	0.013662	0.012873
20	$\mu = 3\pi/2$	-0.010819	0.024488	0.024372	-0.046634	0.012571	0.010425
	$\sigma = 1$						
n	μ	bias($\hat{\mu}$)	MSE($\hat{\mu}$)	var($\hat{\mu}$)	bias($\hat{\sigma}$)	MSE($\hat{\sigma}$)	var($\hat{\sigma}$)
20	$\mu = \pi/2$	0.040485	0.041239	0.039619	-0.116047	0.028651	0.015485
20	$\mu = \pi$	-0.011355	0.048324	0.048196	-0.034567	0.022914	0.021735
20	$\mu = 3\pi/2$	-0.048217	0.041928	0.039634	-0.114967	0.027798	0.014880
	$\sigma = 1.5$						
n	μ	bias($\hat{\mu}$)	MSE($\hat{\mu}$)	var($\hat{\mu}$)	bias($\hat{\sigma}$)	MSE($\hat{\sigma}$)	var($\hat{\sigma}$)
20	$\mu = \pi/2$	0.089913	0.089289	0.082015	-0.316042	0.108749	0.022450
20	$\mu = \pi$	-0.169623	0.182359	0.155325	-0.090631	0.053609	0.045775
20	$\mu = 3\pi/2$	-0.266762	0.122126	0.057966	-0.287900	0.096059	0.024395

Table 9. *Average values of bias, MSE and var for $\hat{\mu}$ and $\hat{\sigma}$ based on 10000 samples for AWN distribution.*

	$\sigma = 0.75$						
n	μ	bias($\hat{\mu}$)	MSE($\hat{\mu}$)	var($\hat{\mu}$)	bias($\hat{\sigma}$)	MSE($\hat{\sigma}$)	var($\hat{\sigma}$)
50	$\mu = \pi/2$	0.009070	0.010114	0.010032	-0.033098	0.005442	0.004354
50	$\mu = \pi$	-0.000557	0.010599	0.010599	-0.012817	0.005449	0.005285
50	$\mu = 3\pi/2$	-0.009217	0.010188	0.010103	-0.032395	0.005303	0.004261
	$\sigma = 1$						
n	μ	bias($\hat{\mu}$)	MSE($\hat{\mu}$)	var($\hat{\mu}$)	bias($\hat{\sigma}$)	MSE($\hat{\sigma}$)	var($\hat{\sigma}$)
50	$\mu = \pi/2$	0.041710	0.017804	0.016076	-0.097183	0.016003	0.006632
50	$\mu = \pi$	-0.006528	0.019235	0.019193	-0.017351	0.009528	0.009229
50	$\mu = 3\pi/2$	-0.051489	0.018352	0.015718	-0.097727	0.016353	0.006857
	$\sigma = 1.5$						
n	μ	bias($\hat{\mu}$)	MSE($\hat{\mu}$)	var($\hat{\mu}$)	bias($\hat{\sigma}$)	MSE($\hat{\sigma}$)	var($\hat{\sigma}$)
50	$\mu = \pi/2$	0.107763	0.043616	0.032259	-0.312824	0.103347	0.010400
50	$\mu = \pi$	-0.156893	0.086969	0.062846	-0.064014	0.025300	0.021279
50	$\mu = 3\pi/2$	-0.230067	0.080164	0.028144	-0.295953	0.095770	0.012167

Table 10. χ^2- *test statistic calculation for AWN distribution.*

Class	O_i	E_i	$\frac{(O_i-E_i)^2}{E_i}$
0-60	74	76	0.03
60-120	74	74	0.00
120-180	97	91	0.43
180-240	106	103	0.08
240-300	84	95	1.22
300-360	71	68	0.13
Total	506	506	1.89

Table 11. χ^2- *test statistic calculation for WN distribution.*

Class	O_i	E_i	$\frac{(O_i-E_i)^2}{E_i}$
0-60	74	68	0.47
60-120	74	78	0.24
120-180	97	94	0.07
180-240	106	100	0.32
240-300	84	90	0.44
300-360	71	74	0.15
Total	506	506	1.69

3. Real data set used

Month	Number of cases
January	40
Febuary	34
March	30
April	44
May	39
June	58
July	51
August	55
September	36
October	48
November	33
December	38
Total	506

Figure 1. *Month of onset of cases of lymphatic leukemia in the UK, 1946-1960.*